

Phase Space Geometry and Chaotic Attractors in Dissipative Nambu Mechanics

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Abstract. Following the Nambu mechanics framework we demonstrate that the non dissipative part of the Lorenz system can be generated by the intersection of two quadratic surfaces that form a doublet under the group $SL(2, \mathbb{R})$. All manifolds are classified into four distinct classes; parabolic, elliptical, cylindrical and hyperbolic. The Lorenz attractor is localized by a specific infinite set of one parameter family of these surfaces. The different classes correspond to different physical systems. The Lorenz system is identified as a charged rigid body in a uniform magnetic field with external torque and this system is generalized to give new strange attractors.

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1. Introduction

We discuss the geometric structure of the integrable part of a chaotic three dimensional dynamical system. Our approach is generic, however we concentrate on the Lorenz model [1]. The analysis is performed in the Nambu mechanics [2] formalism. We find that important information regarding the boundary of the strange attractor can be extracted from this geometric structure. In addition, the symmetry that underlies the phase space geometry leads to mappings between different dynamical systems. Let us review some basic concepts.

In [2] Y. Nambu introduced a formalism of classical mechanics to three dimensional phase spaces. In his words “the Liouville theorem is taken as the guiding principle”. Hence, the formalism applies to systems that preserve the phase space volume, i.e. $\nabla \vec{v} = 0$. The motion of points $\vec{x} = (x, y, z) \in \mathbb{R}^3$ in phase space is determined by two functions of (x, y, z) , called Nambu Hamiltonians H_1 and H_2 , by the equations:

$$\dot{\vec{x}} = \nabla H_1 \times \nabla H_2 \quad (1)$$

By definition (1) the flow is volume preserving, since $\nabla \vec{v} = 0$, $\vec{v} = \dot{\vec{x}}$. In general any function $F(x, y, z)$ is evolving according to

$$\dot{F} = \nabla F \cdot (\nabla H_1 \times \nabla H_2) = \varepsilon_{ijk} \partial_i F \partial_j H_1 \partial_k H_2 \quad (2)$$

where ε_{ijk} is the Levi-Civita tensor. The right-hand side defines the Nambu three bracket:

$$\{F, H_1, H_2\} = \varepsilon_{ijk} \partial_i F \partial_j H_1 \partial_k H_2 \quad (3)$$

The Nambu Hamiltonians are conserved quantities as one can see from (2). The phase space orbit can therefore be realized as the intersection of the surfaces $H_1 = H_1(t=0)$ and $H_2 = H_2(t=0)$.

In [3, 4] the Nambu bracket is realized as the Poisson bracket, denoted as $\{\cdot, \cdot\}_{H_2}$, induced on the manifold $H_2 = H_2(t=0)$, which is now upgraded to a phase space. We will call this bracket Nambu-Poisson bracket and is defined as in (3). It is antisymmetric and satisfies the Jacobi identity. The Nambu-Poisson bracket of one dynamical variable x_i and H_1 is:

$$\{x_i, H_1\}_{H_2} = \varepsilon_{qjk} \partial_q x_i \partial_j H_1 \partial_k H_2 = \varepsilon_{ijk} \partial_j H_1 \partial_k H_2 \equiv \nabla H_1 \times \nabla H_2$$

The dynamical variables form an algebra on H_2

$$\{x_i, x_j\}_{H_2} = \varepsilon_{qpk} \partial_q x_i \partial_p x_j \partial_k H_2 = \varepsilon_{ijk} \partial_k H_2 \quad (4)$$

The evolution of the system is given by

$$\dot{x}_i = \{x_i, H_1\}_{H_2} \quad (5)$$

The surface $H_2(x, y, z) = \text{const.} = H_2(t=0)$ can in some cases be regarded as the group manifold of a Lie group, for example if $H_2 = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$. In this specific example H_2 is just the Casimir of $SO(3)$, so it is reasonable to call it C and we will do so often in this work, calling H_1 as just H as well (in some cases we will stick to H_1, H_2 notation). Now, C defines an algebra of generalized coordinates x, y, z by equation (4):

$$\begin{aligned} \{x, y\}_C &= z \\ \{y, z\}_C &= x \\ \{z, x\}_C &= y \end{aligned}$$

The evolution lies on surface $C = C(t = 0) = \text{const.}$, with dynamics determined by H , or equivalently one can think of the operator $\{\bullet, H\}_C$ as generating the flow by repeated operations of a group element of the group with Casimir C (which group element is determined by H). Therefore, in this case, the surface on which the system is constrained is a group manifold.

There is a freedom [2] in determining H_1, H_2 since any transformation $(H'_1, H'_2) = (H'_1(H_1, H_2), H'_2(H_1, H_2))$ with Jacobian determinant equal to one leaves the equations of motion invariant:

$$\begin{aligned}
\nabla H'_1 \times \nabla H'_2 &= \varepsilon_{ijk} \partial_j H'_1 \partial_k H'_2 \\
&= \varepsilon_{ijk} \left(\frac{\partial H'_1}{\partial H_1} \partial_j H_1 + \frac{\partial H'_1}{\partial H_2} \partial_j H_2 \right) \left(\frac{\partial H'_2}{\partial H_1} \partial_k H_1 + \frac{\partial H'_2}{\partial H_2} \partial_k H_2 \right) \\
&= \varepsilon_{ijk} \left(\frac{\partial H'_1}{\partial H_1} \frac{\partial H'_2}{\partial H_2} - \frac{\partial H'_1}{\partial H_2} \frac{\partial H'_2}{\partial H_1} \right) \partial_j H_1 \partial_k H_2 \\
&= \left| \frac{\partial(H'_1, H'_2)}{\partial(H_1, H_2)} \right| \nabla H_1 \times \nabla H_2
\end{aligned} \tag{6}$$

We will call this freedom a gauge freedom. In the case of linear transformations the gauge group is $SL(2, \mathbb{R})$ and therefore the Nambu Hamiltonians form a doublet $h = (H_1, H_2)$ that transforms under $SL(2, \mathbb{R})$.

In this work we will use a decomposition of the Lorenz system (7), which has three phase space dimensions, to a conservative (non-dissipative) part and a dissipative part [5]. In Hamiltonian mechanics (even phase space dimensions) it is common to add to some known Hamiltonian physical system, forced and dissipative terms. However, it is rarely the case that given the equations of motion of a forced, dissipative system ($\nabla \vec{v} \neq 0$), one is asked to guess a decomposition in a Hamiltonian (non-dissipative) part and a forced, dissipative part. This decomposition is of course not unique, but in Hamiltonian mechanics it is often simple and natural to guess ‘the most physical’ decomposition. For example, suppose one is given the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega x_1 - b x_2
\end{aligned}$$

which is a dissipative system ($\nabla \vec{v} = -b$). It may be decomposed to a non-dissipative part \vec{v}_{ND} with $\nabla \vec{v}_{ND} = 0$ and a dissipative part \vec{v}_D :

$$\vec{v} = \vec{v}_{ND} + \vec{v}_D$$

One may choose the parts $\vec{v}_{ND} = (0, -\omega x_1)$ and $\vec{v}_D = (x_2, -b x_2)$ or even $\vec{v}_{ND} = (x_2 + x_2^2, -\omega x_1)$ and $\vec{v}_D = (-x_2^2, -b x_2)$. Mathematically they are nice equivalent choices but as far as physics is concerned ‘the most physical’ choice would have been $\vec{v}_{ND} = (x_2, -\omega x_1)$ to get the harmonic oscillator for the non-dissipative part and to get the friction force $F = -b \times (\text{velocity})$ for the remaining dissipative part $\vec{v}_D = (0, -b x_2)$.

The situation is analogous in three phase space dimensions using Nambu mechanics, even though one is not guided by intuition to find a physical decomposition. The first time a decomposition in Nambu mechanics was applied was in [5] (to the Lorenz system). In [6] has been set a general framework of the dissipative Nambu mechanics with more examples. The decomposition is chosen in [6] so as $\nabla \vec{v}_{ND} = 0$ and $\nabla \times \vec{v}_D = 0$. It is not unique, however one is restricted by convenience according to motivations. The velocity field can always, at least locally, (see [6] and references therein) be written as

$$\vec{v} = \nabla \times \vec{A} + \nabla D \quad , \quad \text{with} \quad \vec{v}_{ND} = \nabla \times \vec{A} \quad \text{and} \quad \vec{v}_D = \nabla D$$

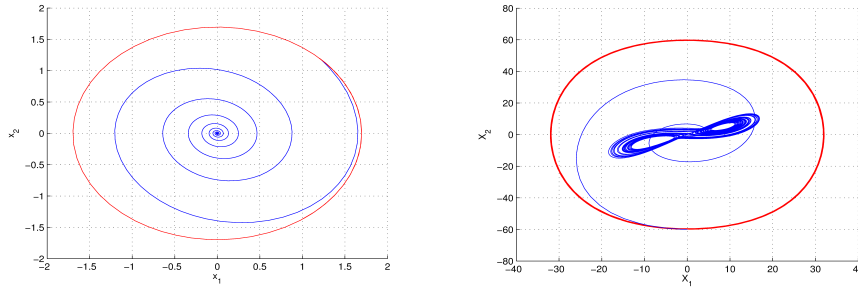
for some vector field \vec{A} and some scalar field D . The field \vec{A} has a gauge degree of freedom $\alpha(\vec{x})$

$$\vec{A} \rightarrow \vec{A} + \nabla\alpha$$

Any vector field can be written $\vec{A} = \nabla\alpha + \beta\nabla\gamma$ [8] for some scalar functions α, β, γ . Working with specific Hamiltonians H_1 and H_2 means that you have chosen the gauge $\vec{A} = H_1\nabla H_2$ for $\beta = H_1$ and $\gamma = H_2$. This gauge is known as the ‘Clebsch-Monge’ gauge and the scalar functions α, β, γ are usually called ‘Monge’s potentials’ (see also [7, 9] for more details on the application of Nambu mechanics on non-Hamiltonian chaos and the quantum, non-commutative approach).

We are interested, and will focus in this work, in the gauge freedom $SL(2, \mathbb{R})$ of H_1 and H_2 . The application is on the Lorenz system, due to its unique significance regarding chaotic dynamics and the interesting physical properties of its non-dissipative part in the specific decomposition we will use. We classify the ‘Nambu doublets’, i.e. the pair of Hamiltonians, of the Lorenz non-dissipative part with criterion the isomorphism of the algebras. We find four different classes of doublets in section 2 that are identified with four different systems in section 4, which are therefore dynamically equivalent. Three of them are very familiar physical systems, namely the simple pendulum, the Duffing oscillator and a charged dielectric rigid body inside a homogeneous magnetic field, with the fourth being an $SO(2, 1)$ system. We may say that these four systems are ‘dual’ to each other.

We stress that Lorenz defines his system as a forced, dissipative system [1]. He starts from an abstract, conservative system with some conservative quantity Q which can be a compact ‘shell like’ surface. Then he makes the hypothesis that ‘[...], whenever Q equals or exceeds some fixed value Q_1 , the dissipation acts to diminish Q more rapidly than the forcing can increase Q , then $(-dQ/dt)$ has a positive lower bound where $Q \geq Q_1$, and each trajectory must ultimately become trapped in the region where $Q < Q_1$.’ quoted straight from the famous 1963 paper [1]. This reasoning



(a) The phase space trajectory of a damped harmonic oscillator. With red the non-dissipative trajectory (harmonic oscillator) corresponding to the initial conditions.

(b) The Lorenz system for large initial (X_1, X_2, X_3) projected on $X_1 - X_2$ plane. The spiral is evolving towards decreasing X_3 . With red the non-dissipative trajectory corresponding to the initial conditions.

Figure 1. The damped harmonic oscillator and the Lorenz system. In the Lorenz system, despite of being a dissipative system ($\nabla\vec{v} < 0$), there is a constant driven term that appears in specific coordinates. For large X_i 's, however, the Lorenz system behaves as a purely dissipative (not driven) system.

implies that the conserved quantities of the corresponding conservative system, which in our decomposition are the Nambu functions[‡], is most natural to be used for finding trapping surfaces and for the localization of the attractor in general. This is what we do in section 3 using the Nambu functions of section 2.

In section 4.3 we find that the Lorenz system is equivalent to a physical system that presents this type of behaviour, quoted earlier, for the forced, damped terms. This system is the $SO(3)$ system we mentioned (charged rigid body in a magnetic field), but now with external torques. The generalized coordinates are the three components of the angular momentum \vec{L} . The conservative function “ Q ” of Lorenz is just the measure of angular momentum $Q = |\vec{L}|$ and its conservation for the conservative part expresses the conservation of angular momentum. The damped terms are torques proportional to angular momentum, like air resistance. There is a constant torque along Z -direction that acts as a driven(forced) term when $L_z < 0$. For large \vec{L} the friction terms dominate over the constant torque (see Figure 1). The attractor in this coordinates lies in $L_z < 0$. Since there exists a constant driven term for L_z , the L_z angular momentum could not ever become constantly zero.

This system (40) is generalized for arbitrary direction of the magnetic field to give a new dynamical system. This general system (44) presents new strange attractors given in Figures 11, 12.

2. Lorenz non dissipative part and $SL(2, \mathbb{R})$ classification

The Lorenz system [1], defined by:

$$\begin{aligned}\dot{X} &= \sigma Y & -\sigma X \\ \dot{Y} &= -XZ + rX & -Y \\ \dot{Z} &= XY & -bZ\end{aligned}\tag{7}$$

where r , σ and b are positive constants[§] is a dissipative system, since $\nabla \vec{v} = -(\sigma + 1 + b) < 0$ where $\vec{v} = (\dot{X}, \dot{Y}, \dot{Z})$. In [5] the following decomposition is proposed:

$$\vec{v}_{ND} = (\sigma Y, -XZ + rX, XY), \quad \vec{v}_D = (-\sigma X, -Y, -bZ)\tag{8}$$

with the full system given by $\vec{v} = \vec{v}_{ND} + \vec{v}_D$. In the same work two Nambu Hamiltonians for the non-dissipative part \vec{v}_{ND} are found

$$H = \frac{1}{2}Y^2 + \frac{1}{2}Z^2 - rZ, \quad C = -\frac{1}{2}X^2 + \sigma Z\tag{9}$$

by integration of the equations:

$$\frac{dY}{dZ} = \frac{r - Z}{Y}, \quad \frac{dZ}{dX} = \frac{X}{\sigma}\tag{10}$$

To decouple the parameter dependence between the two parts (non-dissipative and dissipative) and for reasons of convenience for our purposes we introduce the scaling transformation:

$$X = X_1, \quad Y = \sqrt{r/\sigma}X_2, \quad Z = \sqrt{r/\sigma}X_3, \quad \rho = \sqrt{\sigma r}\tag{11}$$

[‡] We sometimes call the Nambu Hamiltonians of the non-dissipative system, just ‘Nambu functions’.

[§] In every figure of the Lorenz system the free parameters are assumed to be $r = 28$, $\sigma = 10$ and $b = 8/3$, unless stated differently.

and the Lorenz system (7) becomes:

$$\begin{aligned}\dot{X}_1 &= \rho X_2 & -\sigma X_1 \\ \dot{X}_2 &= -X_1 X_3 + \rho X_1 & -X_2 \\ \dot{X}_3 &= X_1 X_2 & -bX_3\end{aligned}\quad (12)$$

We will work independently on the non dissipative part:

$$\begin{aligned}\dot{X}_1 &= \rho X_2 \\ \dot{X}_2 &= -X_1 X_3 + \rho X_1 \\ \dot{X}_3 &= X_1 X_2\end{aligned}\quad (13)$$

The Nambu Hamiltonians (9) become in the new rescaled variables:

$$H = \frac{1}{2}X_2^2 + \frac{1}{2}X_3^2 - \rho X_3, \quad C = -\frac{1}{2}X_1^2 + \rho X_3 \quad (14)$$

We will call this pair of functions a ‘Nambu doublet’

$$h = \begin{pmatrix} H \\ C \end{pmatrix} \quad (15)$$

since it transforms as a whole under a restricted class of transformations. These are all transformations with Jacobian determinant equal to one, as we saw in section 1. We will restrict ourselves in linear transformations so as to work only with quadratic functions. In this case, the transformation group is the group of real 2×2 matrices with determinant one, namely the $SL(2, \mathbb{R})$. For example let generate a different doublet by action of an $SL(2, \mathbb{R})$ matrix:

$$h' = Ah \Rightarrow \begin{pmatrix} H' \\ C' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H \\ C \end{pmatrix} = \begin{pmatrix} C \\ -H \end{pmatrix} \quad (16)$$

We see that the role of H and C can be interchanged, as long as you change the sign of one of the functions, since we could have used

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as well^{||}. Another transformation could be:

$$\begin{pmatrix} H_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} H \\ C \end{pmatrix}$$

that gives:

$$\begin{pmatrix} H_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}X_1^2 - \rho X_3 \\ -\frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \frac{1}{2}X_3^2 \end{pmatrix} \quad (17)$$

Recall from the Introduction, that the Nambu Hamiltonians H , C of a volume preserving system define surfaces in phase space because they are conserved. Once initial conditions are given, the solution of the system should lie on both surfaces defined by

$$\begin{pmatrix} H(X_1, X_2, X_3) \\ C(X_1, X_2, X_3) \end{pmatrix} = \begin{pmatrix} H(X_1(0), X_2(0), X_3(0)) \\ C(X_1(0), X_2(0), X_3(0)) \end{pmatrix} \quad (18)$$

Therefore the intersection of the two Hamiltonians defines the phase space trajectory (see Figure 2). Thus, given a Nambu doublet and initial conditions, it is not only given the equations of motion (velocity field) but the solution of the system, as well!

^{||} It is apparent that the system is invariant in these cases since $\dot{X}_i = \nabla H \times \nabla C = (-\nabla C) \times \nabla H = \nabla C \times (-\nabla H)$

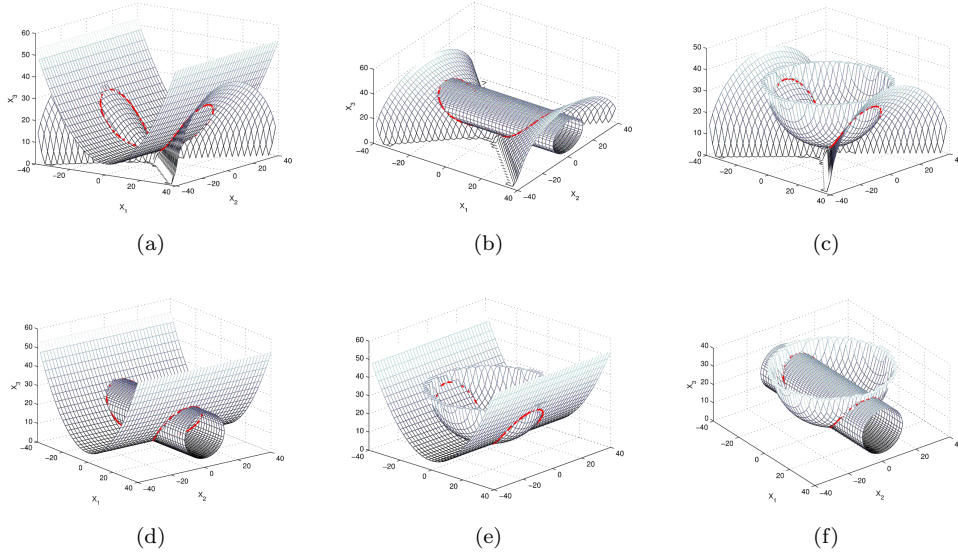


Figure 2. An orbit, in red, of the conservative system (13) as defined by the intersection of various Nambu surfaces (same orbit at each case).

In equation (17) is defined the Nambu Hamiltonian function C_0 that defines a hyperboloid, once initial conditions are given. This is a topologically different surface from the cylinder and parabolic surface of equation (14). An $SL(2, \mathbb{R})$ transformation led to a new type of surface. This is reflected to the algebra of the Nambu bracket, since:

$$\begin{aligned} \{X_1, X_2\}_C &= \rho \\ \{X_2, X_3\}_C &= -X_1 \\ \{X_3, X_1\}_C &= 0 \end{aligned} \quad (19)$$

while:

$$\begin{aligned} \{X_1, X_2\}_{C_0} &= X_3 \\ \{X_2, X_3\}_{C_0} &= -X_1 \\ \{X_3, X_1\}_{C_0} &= X_2 \end{aligned} \quad (20)$$

The later algebra is the $SO(2, 1)$ algebra \P . We could get an isomorphic to $SO(2, 1)$ algebra by applying to (17) transformations of the form

$$A = \begin{pmatrix} \zeta & 0 \\ 0 & 1/\zeta \end{pmatrix} \quad (21)$$

for some $\zeta \in \mathbb{R}$. These algebras would correspond to other hyperboloids:

$$C' = -\frac{1}{2}\frac{1}{\zeta}X_1^2 + \frac{1}{2}\zeta X_2^2 + \frac{1}{2}\zeta X_3^2 - (\zeta - \frac{1}{\zeta})\rho X_3$$

\P Using the algebra (20), the properties of the Poisson bracket and H_0 one can show that the doublet (17) gives the system (13):

$$\begin{aligned} \dot{X}_1 &= \{X_1, H_0\}_{C_0} = \{X_1, \frac{1}{2}X_1^2\}_{C_0} + \{X_1, -\rho X_3\}_{C_0} = \rho X_2 \\ \dot{X}_2 &= \{X_2, H_0\}_{C_0} = \{X_2, \frac{1}{2}X_1^2\}_{C_0} + \{X_2, -\rho X_3\}_{C_0} = -X_1 X_3 + \rho X_1 \\ \dot{X}_3 &= \{X_3, H_0\}_{C_0} = \{X_3, \frac{1}{2}X_1^2\}_{C_0} + \{X_3, -\rho X_3\}_{C_0} = X_1 X_2 \end{aligned}$$

One may ask what other types of quadratic surfaces one may get. We answer this question and classify the doublets according to isomorphisms of the algebras.

Let an arbitrary $SL(2, \mathbb{R})$ matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1$$

act on the doublet (17). Then we get a general expression for the possible quadratic doublets⁺:

$$h = \begin{pmatrix} S(X_1, X_2, X_3; \alpha, \beta) \\ S(X_1, X_2, X_3; \gamma, \delta) \end{pmatrix} \quad \forall \alpha, \beta, \gamma, \delta \in \mathbb{R} : \alpha\delta - \beta\gamma = 1 \quad (22)$$

with

$$S(X_1, X_2, X_3; \alpha, \beta) = (\alpha - \beta) \frac{1}{2} X_1^2 + \beta \frac{1}{2} X_2^2 + \beta \frac{1}{2} X_3^2 - \alpha \rho X_3 \quad (23)$$

Of course it does not matter with what doublet one may begin, since one can always with a re-parametrization end up with equation (23). The expression for $S(X_1, X_2, X_3; \gamma, \delta)$ is identical with 23 and describes exactly the same surfaces. The $S(\alpha, \beta)$ and $S(\gamma, \delta)$ are different only when considered as part of a doublet, because it is then imposed the restriction $\alpha\delta - \beta\gamma = 1$.

	$C_{(\lambda)} = (\lambda - 1) \frac{1}{2} X_1^2 + \frac{1}{2} X_2^2 + \frac{1}{2} (X_3 - \lambda \rho)^2$
$\lambda < 1$: a set of hyperboloids centred at $X_3^c = \lambda \rho$
$\lambda > 1$: a set of ellipsoids centred at $X_3^c = \lambda \rho$
$\lambda = 1$: a cylinder centred at $X_3^c = \rho$
$\lambda \rightarrow \pm\infty$: a parabolic cylinder

Table 1. The Nambu surfaces

The possible types of quadratic surfaces are given by equation (23). These determine the possible Nambu algebras and the geometry of the phase-space through the Nambu bracket:

$$\{X_i, X_j\}_S = \epsilon_{ijk} \partial_k S$$

Let $\lambda = \frac{\alpha}{\beta} \in \mathbb{R}$ and

$$C_{(\lambda)} = (\lambda - 1) \frac{1}{2} X_1^2 + \frac{1}{2} X_2^2 + \frac{1}{2} (X_3 - \lambda \rho)^2 \quad (24)$$

where $C_{(\lambda)}$ is an abbreviation for $C(X_1, X_2, X_3; \lambda)$. Then, up to a constant shift it is:

$$S(X_1, X_2, X_3; \alpha, \beta) = \beta C_{(\lambda)}$$

Rescaling of a Nambu function: $H'_1 = \beta H_1$ leaves the corresponding Nambu surface invariant since: $H'_1 = H'_1(0) \Leftrightarrow \beta H_1 = \beta H_1(0) \Leftrightarrow H_1 = H_1(0)$ so that the equations $H'_1 = H'_1(0)$ and $H_1 = H_1(0)$ define exactly the same surface in the phase space. Therefore equation (24) describes all possible quadratic surfaces, which are listed in

⁺ Note that there does not exist another different distinct set of quadratic doublets. If it existed would mean that members of one set are not related with members of the other set by transformations with Jacobian determinant one, which is not allowed as can be realized by (6). In addition quadratic doublets can only be related with linear $SL(2, \mathbb{R})$ transformations with each other, because any non-linear transformation would raise the rank of the functions.

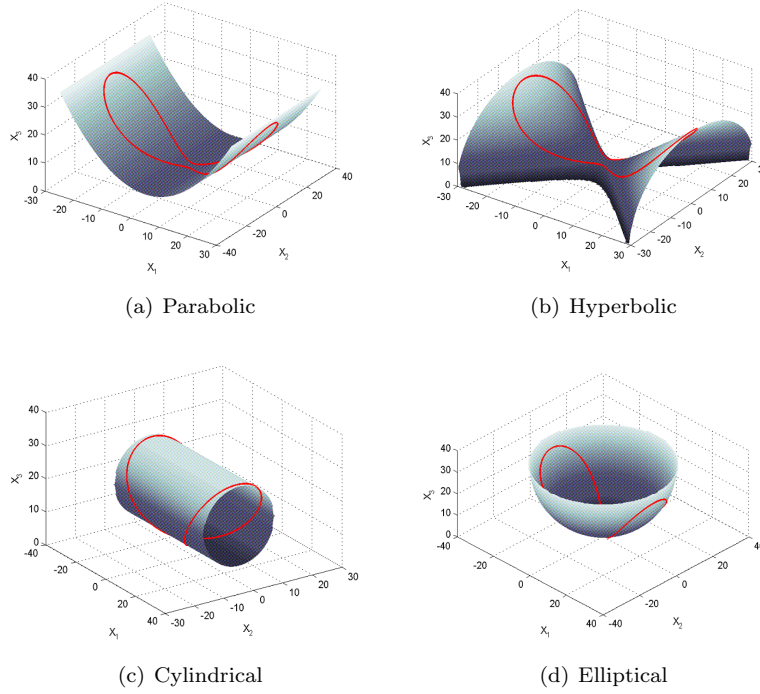


Figure 3. The four different classes of the Nambu surfaces (invariant manifolds) of the non dissipative part (13) of the Lorenz system. An orbit, in red, of the system lies on any of these surfaces.

Table 1 and can be seen in Figure 3. From the algebraic point of view the rescaling leads to isomorphic algebras. We see that there are infinite surfaces, of four types, when classified with respect to the isomorphisms of the corresponding algebras. The surface given by $S(\alpha, 0)$ (that is C of equation (14)) corresponds to $\lambda \rightarrow \pm\infty$.

Let list the possible algebras. The hyperboloid centred at the origin C_0 gives the $SO(2, 1)$ algebra and any other hyperboloid gives an algebra isomorphic to $SO(2, 1)$.

The sphere $C_2 = \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \frac{1}{2}(X_3 - 2\rho)^2$ centred at $X_3^c = 2\rho$, after the transformation $X_3 \rightarrow X_3 - 2\rho$, gives the $SO(3)$ algebra $\{X_i, X_j\}_{C_2} = \varepsilon_{ijk}X_k$, and any ellipsoid, an algebra isomorphic to $SO(3)$.

The cylinder $C_1 = \frac{1}{2}X_2^2 + \frac{1}{2}(X_3 - \rho)^2$ gives the $SE(2)$ algebra after the transformation $X_3 \rightarrow X_3 - \rho$.

The algebra obtained by the parabolic cylinder $C_{(\infty)}$ call it $C = -\frac{1}{2}X_1^2 + \rho X_3$ can be realized as the Heisenberg algebra $\{X_1, X_2\}_C = \rho$ of two independent variables X_1, X_2 on the curved plane C : $\{X_2, X_3\}_C = -X_1$, $\{X_3, X_1\}_C = 0$.

With these four types we classify the possible Nambu doublets as in Table 2 and an orbit defined by six different doublets can be seen in Figure 2. In Table 2 one of the four possible types of algebras is chosen with a specific C at each case and the corresponding H is found by the condition $\alpha\delta - \beta\gamma = 1$.

	$H = (\alpha - 1)\frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \frac{1}{2}X_3^2 - \alpha\rho X_3, \forall \alpha \in \mathbb{R}$ $\begin{aligned} \{X_1, X_2\}_C &= \rho \\ \{X_2, X_3\}_C &= -X_1 \\ \{X_3, X_1\}_C &= 0 \end{aligned}$
	$H = (1 - \beta)\frac{1}{2}X_1^2 + \beta\frac{1}{2}X_2^2 + \beta\frac{1}{2}X_3^2 - \rho X_3, \forall \beta \in \mathbb{R}$ $SO(2, 1) \quad \begin{aligned} \{X_1, X_2\}_{C_0} &= X_3 \\ \{X_2, X_3\}_{C_0} &= -X_1 \\ \{X_3, X_1\}_{C_0} &= X_2 \end{aligned}$
	$H = \frac{1}{2}X_1^2 + \beta\frac{1}{2}X_2^2 + \beta\frac{1}{2}X_3^2 - \rho(\beta + 1)X_3, \forall \beta \in \mathbb{R}$ $SE(2) \quad \begin{aligned} \{X_1, X_2\}_{C_1} &= X_3 - \rho \\ \{X_2, X_3\}_{C_1} &= 0 \\ \{X_3, X_1\}_{C_1} &= X_2 \end{aligned}$
	$H = (\beta + 1)\frac{1}{2}X_1^2 + \beta\frac{1}{2}X_2^2 + \beta\frac{1}{2}X_3^2 - \rho(2\beta + 1)X_3, \forall \beta \in \mathbb{R}$ $SO(3) \quad \begin{aligned} \{X_1, X_2\}_{C_2} &= X_3 - 2\rho \\ \{X_2, X_3\}_{C_2} &= X_1 \\ \{X_3, X_1\}_{C_2} &= X_2 \end{aligned}$

Table 2. The Nambu doublets Classification

3. Localization of the Lorenz Attractor

When the dissipative part $v_D = (-\sigma X_1, -X_2, -bX_3)$ is added to the system (13), the Lorenz attractor is formed. This is a global attractor, that is this subset of phase space that every trajectory of any initial conditions approaches for $t \rightarrow \infty$. By definition the attractor should lie in a bounded region and determining this region is what we call localization. The possible geometries of the boundary of the Lorenz attractor have been discussed by Lorenz [1, 10], Sparrow [11] and others [12, 13]. We will not find new surfaces, not mentioned in bibliography already. Our scope is only to prove that the Nambu functions found in previous section, i.e. the quadratic invariant manifolds of the non-dissipative part, define proper localization surfaces.

Let us change the notation and use just S for $2C_{(\lambda)}$ of equation (24).

$$S(X_1, X_2, X_3) = (\lambda - 1)X_1^2 + X_2^2 + (X_3 - \lambda\rho)^2 \quad (25)$$

When X_1, X_2, X_3 are subject to the Lorenz evolution (with the dissipative part added), S is not conserved but is changing with time:

$$\dot{S} = \frac{1}{2}b(\lambda\rho)^2 - 2 \left(\sigma(\lambda - 1)X_1^2 + X_2^2 + b(X_3 - \frac{1}{2}\lambda\rho)^2 \right) \quad (26)$$

Assume for the moment $\lambda > 1$. Suppose there exists some real constant K_{max} for which \dot{S} can be positive only inside the ellipsoid $S = K_{max}$ and nowhere else (see Appendix C of Sparrow [11] and Doering and Gibbon [12], as well). Then for any point outside this ellipsoid, $\dot{S} < 0$ and therefore S will be decreasing until it gets $S < K_{max}$ i.e. the trajectory gets inside the ellipsoid. Of course, if the trajectory starts inside this ellipsoid, it will remain there, since on the outside is everywhere $\dot{S} < 0$. If such K_{max} exists, it is the solution of the constrained maximization problem:

$$K_{max} = \max\{S(X_1, X_2, X_3) : \dot{S} \geq 0\} \quad , \lambda \geq 1 \quad (27)$$

This is a typical problem to solve with the Lagrange multipliers method (Appendix A). This condition defines a compact attracting surface (in fact every other ellipsoid

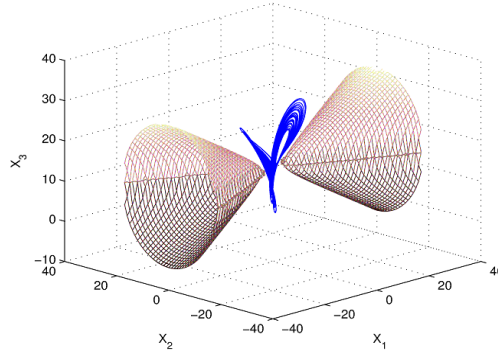


Figure 4. The repelling two-sheeted hyperboloid $\lambda = 0.8$ and the Lorenz attractor.

$S = K$ with $K > K_{max}$ is an attracting surface, since $\dot{S} < 0$ everywhere on the outside of all of these). The solution of (27) is given in (30) where $K_{max} = R_a^2$.

We argue here that similar reasoning can be applied for non-positive-definite functions (non-compact surfaces). For $\lambda < 1$ the surfaces are hyperboloids which are non compact. In this case the equation $S = K$ for some constant K defines a one sheeted hyperboloid if $K > 0$ and a two sheeted hyperboloid if $K < 0$. For simplicity, with no loss of generality, assume that $K < 0$. Then the condition $S > K$ defines the region outside the ‘lobes’ of the two sheeted hyperboloid (the attractor lies in this region in Figure 4), while $S < K$ defines the region inside the lobes. As previously, assume there exists some constant value K_{min} such that \dot{S} can be negative only outside the lobes (region $S > K_{min}$) and nowhere else. Then any trajectory starting from inside the lobes will be expelled to the outside, since $\dot{S} > 0$ in the inside and therefore S is increasing until it gets $S > K_{min}$ i.e. the trajectory gets outside the lobes. Of course, if it starts outside the lobes, it is impossible to get inside. We call such a kind of surface, a repelling surface. The same reasoning can be applied to any non-compact surface. The problem is now a minimization problem*

$$K_{min} = \min\{S(X_1, X_2, X_3) : \dot{S} \leq 0\} \quad , \lambda < 1 \quad (28)$$

The solution is given in (30) where $K_{min} = -R_r^2$. In the Appendix A is given analytically the Lagrange’s multipliers method.

Another way to understand localization is to think of a fixed surface and imagine the flow at each point crossing it (Giacomini and Neukirch [13]). For a localization surface the flow should cross every point of the surface towards the same direction (semi-permeable surface) [13]; inwards for the attracting (compact) surfaces and outwards for the repelling (non-compact) ones. Or else the trajectory could pass through the surface in one direction at some point and get through the opposite direction at some other point of the surface. Thus, the scalar product of the vector normal to the surface ∇S at each point and the vector tangent to the flow \vec{v} should have a constant sign at each point. That is:

$$\dot{S} = (\nabla S) \cdot \dot{\vec{X}} \quad (29)$$

* The signs is a matter of convention, since we could have used a function $\tilde{S} = -S$ and turn the problem to a maximization problem.

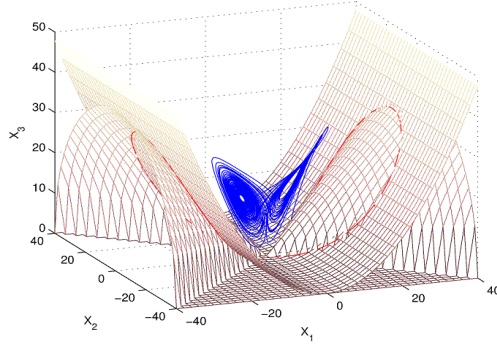


Figure 5. The homoclinic orbits of the conservative system (13) with red as defined by the repelling cone and parabolic cylinder. When the dissipative part is added, the flow is forced towards the Lorenz attractor.

has constant sign at every point on a localization surface. This is proven in the Appendix B for all repelling surfaces of Equation (31).

Let summarize our results. We find the following attracting and repelling surfaces:

$$\begin{aligned}
 S(X_1, X_2, X_3) = K &\Leftrightarrow (\lambda - 1)X_1^2 + X_2^2 + (X_3 - \lambda\rho)^2 = K \\
 \forall K \in \mathbb{R} : K &\begin{cases} \geq R_a^2 & \text{for } \lambda \geq 1 \\ \leq \tilde{R}_r^2 & \text{for } \{\lambda = 0, -\infty\} \text{ and } \{\lambda = (-\infty, 0), X_3 > 0\} \\ \leq -R_r^2 & \text{for } \lambda < 1 \end{cases} \quad (30)
 \end{aligned}$$

where: $R_a = \begin{cases} \lambda\rho & \text{for } b \leq 2 \\ \lambda\rho \frac{b}{2\sqrt{b-1}} & \text{for } b > 2 \end{cases}$, $\tilde{R}_r = \lambda\rho$, $R_r = \lambda\rho \frac{b}{2\sqrt{\sigma(\sigma-b)}}$

The attracting cases hold for $\sigma > 1$, the repelling cases hold for $\sigma > 1$ and $\sigma > b$ except for the parabolic surface $\lambda \rightarrow -\infty$ that is $-X_1^2 + 2\rho X_3 = 0$ which holds for $\sigma > b/2$. The surfaces that are closer to the attractor and provide the optimum localization are the extreme cases of (30) that correspond to the equalities:

$$S = R_a^2, S = \tilde{R}_r^2, S = -R_r^2 \quad (31)$$

This means that the strange attractor lies in the region \mathcal{A} that is defined as:

$$\mathcal{A} = \{(X_1, X_2, X_3) : S \leq R_a^2 \text{ and } S \geq \tilde{R}_r^2 \text{ and } S \geq -R_r^2\} \quad (32)$$

The following surfaces of (30) can be found in Doering and Gibbon [12]: the repelling parabolic cylinder $-X_1^2 + 2\rho X_3 = 0$ ($\lambda \rightarrow -\infty$), the repelling cone $-X_1^2 + X_2^2 + X_3^2 = 0$ ($\lambda = 0$), the attracting cylinder $X_2^2 + (X_3 - \rho)^2 = \rho^2$ ($\lambda = 1$) and the attracting sphere $X_1^2 + X_2^2 + (X_3 - 2\rho)^2 = (2\rho)^2$ ($\lambda = 2$). The two-sheeted hyperboloids ($\lambda < 1$) (see Figure 4) were for the first time found by Giacomini and Neukirch [13]. In Figure 5 we see that the repelling cone and the repelling parabolic surface define the homoclinic orbits of the non-dissipative system. For $b \leq 2$ these homoclinic orbits define also the attracting surfaces (Figure 6).

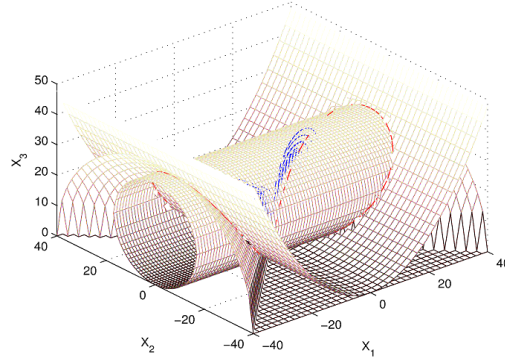


Figure 6. For $b \leq 2$ the strange attractor is localized by the corresponding surfaces of the homoclinic orbit. Here for $b = 2$ are shown just three of these surfaces. The strange attractor is fairly seen in blue in the interior and the homoclinic orbit is shown in red.

4. Correspondence with physical systems

The different Nambu doublets can be interpreted as different physical systems. The one Nambu Hamiltonian is regarded to be the energy of the system and the other has different interpretations depending on the case (a constraint of the motion or an integral of motion). This framework is providing a way to identify completely different physical systems that have the same dynamics. From the mathematical point of view it is just one system in different representations. In our case these are the Heisenberg, $SE(2)$, $SO(3)$ and $SO(2,1)$ formulations that are linked with each other.

4.1. The unforced, undamped Duffing oscillator

Let us work with the first doublet (H, C) of Table 2. We reduce the system on the parabolic cylinder $C = -\frac{1}{2}X_1^2 + \rho X_3$ and the Hamiltonian becomes:

$$H = \frac{1}{2}X_2^2 - \frac{1}{2\rho^2}(\rho^2 - C)X_1^2 + \frac{1}{8\rho^2}X_1^4 + \text{const.}$$

We see that $\forall \lambda$ the Hamiltonian is the same up to a constant shift. Let ρ be absorbed in the Hamiltonian, then

$$H = \frac{1}{2}\rho X_2^2 - \frac{1}{2\rho}(\rho^2 - C)X_1^2 + \frac{1}{8\rho}X_1^4 \quad (33)$$

Our dynamical variables are now the conjugate canonical variables (X_1, X_2) and the Nambu bracket reduces to Poisson bracket on a $2D$ phase space (Figure 7).

$$\{X_1, X_2\}_C = 1$$

The system (33) describes a particle moving along X_1 with momentum X_2 under the influence of the potential: $V(X_1) = -\frac{1}{2\rho}(\rho^2 - C)X_1^2 + \frac{1}{8\rho}X_1^4$ where $C = \text{const.}$ depends on the initial conditions. The equilibria depend on the value of C . They are $\bar{X} = (0, 0)$

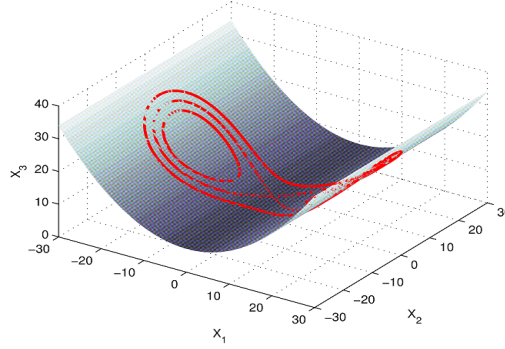


Figure 7. The phase space portrait of the system (13) reduced on the parabolic cylinder C . Choosing initial conditions for the system (13) is equivalent to choosing not only initial conditions but also a potential for the system (33). The value of the potential is determined by the level of the surface C .

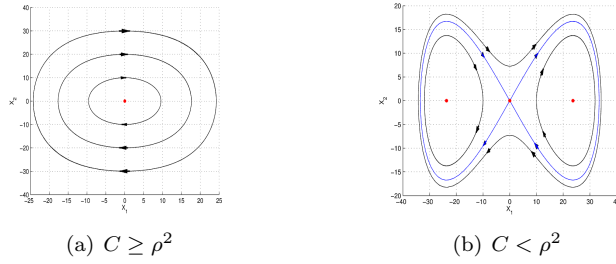


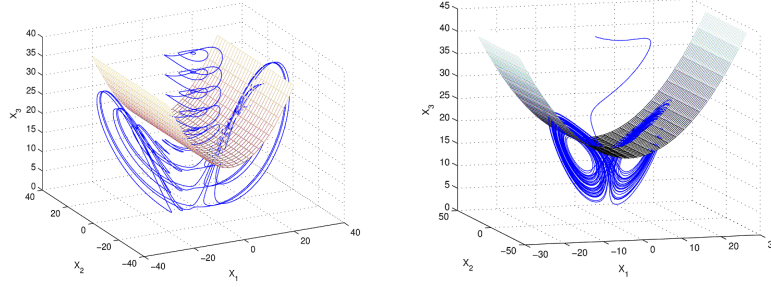
Figure 8. The phase space of system (33). It depends on the initial conditions of system (13) and specifically on $X_3(t=0)$ through the value of the function C .

for $C \geq \rho^2$ and

$$\bar{X} = \begin{cases} (0, 0) \\ (-\sqrt{2(\rho^2 - C)}, 0) \\ (\sqrt{2(\rho^2 - C)}, 0) \end{cases} \quad \text{for } C < \rho^2$$

This system is the unforced, undamped Duffing oscillator $\ddot{x} = -\alpha x - \beta x^3$ for $\alpha = C - \rho^2$ and $\beta = \frac{1}{2}$. For $C > \rho^2$ the potential is a single well while for $C < \rho^2$ the potential is a double well and the symmetry of the vacuum state is ‘spontaneous broken’ (Figure 8).

This simple analogue of a mechanism of ‘spontaneous symmetry breaking’ gives an intuitive picture of the formation of the attractor. Equation $C = \rho^2$ defines a critical parabolic surface that separates phase space in the upper region of one vacuum state and the lower region of two vacuum states (Figure 9(a)). Suppose the initial conditions are above that surface and dissipation is “turned on”. The system will flow towards the origin until it passes the critical surface when it enters the broken symmetry region and has to choose between different vacuum states. It may oscillate around a while but eventually it is forced outside the critical surface again. The procedure is constantly



(a) The 3D phase space of the non dissipative part (13) and the critical parabolic surface.

(b) The Lorenz system for initial conditions $\vec{X} = (0, 20, 40)$ and the critical parabolic surface.

Figure 9. The resemblance with a ‘spontaneous symmetry breaking mechanism’ gives an intuitive explanation for the formation of the attractor.

repeated and the attractor is formed (see Figure 9).

This is a qualitative picture. However, when the dissipative part is added, the potential becomes another independent variable of the system. With the transformation $\tilde{X}_2 = X_2 - \frac{\sigma}{\rho}X_1$ the full Lorenz system becomes:

$$\begin{aligned}\dot{X}_1 &= \rho X_2 \\ \dot{\tilde{X}}_2 &= -\frac{1}{2\rho}X_1^3 + \frac{1}{\rho}(\rho^2 - \sigma - C)X_1 - (\sigma + 1)\tilde{X}_2 \\ \dot{C} &= (\sigma - \frac{b}{2})X_1 - bC\end{aligned}$$

We see that a locally time dependent potential cannot be defined. The oscillator develops a friction term and we obtain the Takeyama [14, 15] memory term in the potential: $C(t) = C(0)e^{-bt} + (\sigma - \frac{b}{2})e^{-bt} \int_0^t d\xi e^{b\xi} X_1^2(\xi)$.

4.2. Simple pendulum

Now, let work on the Nambu doublet of Table 2 corresponding to the cylinder $C_1 = \frac{1}{2}X_2^2 + \frac{1}{2}(X_3 - \rho)^2$. Let apply the transformation $\tilde{X}_3 = X_3 - \rho$. The non dissipative part (13) becomes:

$$\vec{v}_{ND} = (\rho X_2, -X_1\tilde{X}_3, X_1X_2) \quad (34)$$

and the full Lorenz system becomes:

$$\vec{v} = (\rho X_2 - \sigma X_1, -X_1\tilde{X}_3 - X_2, X_1X_2 - b\tilde{X}_3 - b\rho) \quad (35)$$

The Lorenz system is just translated along X_3 axis with the equilibrium of the origin translated at $\tilde{X}_3 = -\rho$. Let study independently the volume preserving part (34). The cylinder becomes: $C_1 = \frac{1}{2}X_2^2 + \frac{1}{2}\tilde{X}_3^2$ and we choose the Hamiltonian for $\beta = 0$:

$$H = \frac{1}{2}X_1^2 - \rho\tilde{X}_3 \quad (36)$$

Note the analogy of this Hamiltonian to the one of a simple pendulum $H_{pend.} = \frac{1}{2}p_\theta^2/(ml^2) - mgl\cos\theta$. We perform the following transformations:

$$X_1 = -p_\theta, \quad X_2 = R\sin\theta, \quad \tilde{X}_3 = R\cos\theta$$

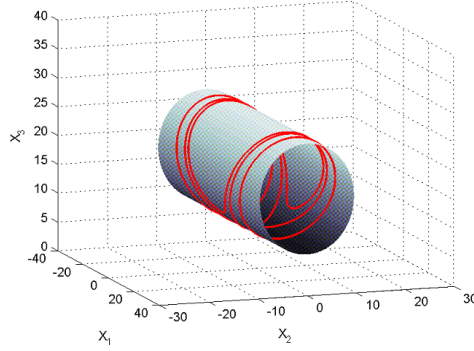


Figure 10. The phase space portrait of the system (13) reduced on the cylinder C_1 . Choosing initial conditions for the system (13) is equivalent to choosing both initial conditions and the length for a simple pendulum. The length is determined by the radius of the cylinder.

where our new variables are (θ, p_θ) since: $R = \sqrt{2C_{(1)}} = \text{const.}$ We see that C_1 is just expressing the constraint that the simple pendulum has constant length. In fact we have parametrized C_1 with (θ, p_θ) : $X_i = X_i(\theta, p_\theta)$. We have: $\theta = \tan^{-1} \frac{X_2}{X_3}$ and therefore

$$\frac{\partial \theta}{\partial X_1} = 0, \quad \frac{\partial \theta}{\partial X_2} = \frac{\tilde{X}_3}{2C_1}, \quad \frac{\partial \theta}{\partial \tilde{X}_3} = -\frac{X_2}{2C_1}$$

Using these derivatives is straightforward to calculate the following Nambu-Poisson brackets:

$$\{\theta, p_\theta\}_{C_1} = 1, \quad \{\theta, H\}_{C_1} = \frac{\partial H}{\partial p_\theta}, \quad \{p_\theta, H\}_{C_1} = -\frac{\partial H}{\partial \theta}$$

Hence, the canonical equations are retrieved with:

$$H = \frac{1}{2}p_\theta^2 - \rho R \cos \theta$$

Thus, Lorenz volume preserving motion (34) reduces to pendulum motion \sharp on level surfaces of C_1 (Figure 10).

When the dissipative part is added the equations of motion (full Lorenz system) become:

$$\begin{aligned} R\dot{\theta} &= Rp_\theta - (1-b)R \sin \theta \cos \theta - b\rho \sin \theta \\ \dot{p}_\theta &= -\rho R \sin \theta - \sigma p_\theta \\ \dot{R} &= -b\rho \cos \theta - R(\sin^2 \theta + b \cos^2 \theta) \end{aligned} \quad (37)$$

4.3. Charged rigid body in a uniform magnetic field

Let us work on the Nambu doublet of $C_2 = \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \frac{1}{2}(X_3 - 2\rho)^2$ with the transformations

$$L_1 = X_1, \quad L_2 = X_2, \quad L_3 = X_3 - 2\rho \quad (38)$$

\sharp An analogous correspondence between the simple pendulum and the free rigid body can be found in [16]

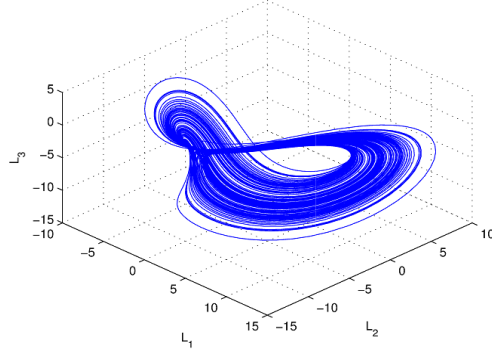


Figure 11. The phase-space of system (44) for $I_1 = 1/8$, $I_2 = 1/3$, $I_3 = 1/2$, $\rho_1 = 0$, $\rho_2 = 22$, $\rho_3 = 30$, $\sigma = 10$, $b = 1$ and initial conditions $\vec{L}_o = (3, -4, -10)$.

The non dissipative part becomes:

$$\vec{v}_{ND} = (\rho L_2, -L_1 L_3 - \rho L_1, L_1 L_2) \quad (39)$$

and the full Lorenz system becomes:

$$\begin{aligned} \dot{L}_1 &= \rho L_2 & -\sigma L_1 \\ \dot{L}_2 &= -L_1 L_3 - \rho L_1 & -L_2 \\ \dot{L}_3 &= L_1 L_2 & -b L_3 - 2b\rho \end{aligned} \quad (40)$$

The Lorenz system is just translated along X_3 axis with the equilibrium of the origin translated at $L_3 = -2\rho$. The Nambu function C_2 becomes:

$$C_2 = \frac{1}{2}L_1^2 + \frac{1}{2}L_2^2 + \frac{1}{2}L_3^2 \quad (41)$$

giving the algebra of $SO(3)$. Let work with the corresponding Hamiltonian for $\beta = 1$:

$$H = L_1^2 + \frac{1}{2}L_2^2 + \frac{1}{2}L_3^2 - \rho L_3 \quad (42)$$

We identify L_i as the angular momenta and C_2 is expressing the conservation of $|\vec{L}|$. This system can be realized as a uniformly charged dielectric rigid body with charge Q in a uniform magnetic field $\vec{B} = (0, 0, B)$ with moments of inertia along the primary axis $I_1 = 1/2$ and $I_2 = I_3 = 1$ and $\rho = QB$. The dissipative part is just the external torque: a damped term proportional to angular momentum for each component and a constant term $-2b\rho$ in L_3 component that acts as a driven torque for $L_3 < 0$. For an arbitrary rigid body with any moments of inertia and any direction of the magnetic field the Hamiltonian is:

$$H = H_{free} + H_{int}$$

where

$$\begin{aligned} H_{free} &= \frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}L_2^2 + \frac{1}{2I_3}L_3^2 \\ H_{int} &= -Q\vec{B} \cdot \vec{L} \end{aligned} \quad (43)$$

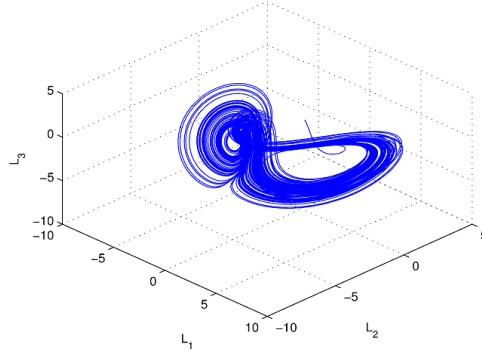


Figure 12. The phase-space of system (44) for $I_1 = 1/20$, $I_2 = 1/10$, $I_3 = 1/5$, $\rho_1 = 10$, $\rho_2 = \rho_3 = 25$, $\sigma = 10$, $b = 1$ and initial conditions $\vec{L}_o = (-1, 1, -1)$.

Assuming $\vec{\rho} = Q\vec{B}$ and adding the external torque $\vec{N}_{ext} = (-\sigma L_1, -L_2, -bL_3 - 2b\rho)$ we get the equations of motion for the general system:

$$\begin{aligned} \dot{L}_i &= \{L_i, H\}_{C_2} + \vec{N}_{ext} \Rightarrow \\ \dot{L}_1 &= \left(\frac{1}{I_2} - \frac{1}{I_3} \right) L_2 L_3 + \rho_3 L_2 - \rho_2 L_3 - \sigma L_1 \\ \dot{L}_2 &= \left(\frac{1}{I_3} - \frac{1}{I_1} \right) L_1 L_3 + \rho_1 L_3 - \rho_3 L_1 - L_2 \\ \dot{L}_3 &= \left(\frac{1}{I_1} - \frac{1}{I_2} \right) L_1 L_2 + \rho_2 L_1 - \rho_1 L_2 - bL_3 - 2b\rho_3 \end{aligned} \quad (44)$$

which can also be written as $\dot{\vec{L}} + \vec{\omega} \times \vec{L} + Q\vec{B} \times \vec{L} = \vec{N}_{ext}$. System (44) presents strange attractors as can be seen in Figure 11 and Figure 12.

4.4. $SO(2,1)$ formulation

The $SO(2,1)$ case can be regarded as a ‘hyperbolic’ analogue of the rigid body $SO(3)$ case. This is mainly of mathematical interest. One gets this hyperbolic analogue with the Nambu doublet ($\beta = 2$ for example)

$$C_0 = -\frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \frac{1}{2}X_3^2, \quad H = -\frac{1}{2}X_1^2 + X_2^2 + X_3^2 - \rho X_3 \quad (45)$$

Depending on the initial conditions the hyperboloid C_0 may be two-sheeted which corresponds to the hyperbolic space \mathbb{H}^2 or one-sheeted, which corresponds to anti-de Sitter (or de Sitter) space AdS_2 (or dS_2). The phase space portrait of system (45) for the AdS_2 case can be seen in Figure 13.

5. Conclusions

The gauge sector of the phase space geometry of a three dimensional volume preserving dynamical system is investigated. As a toy model we used the non dissipative part of the Lorenz system. Searching for possible valuable informations one may retrieve from this geometric picture, we found that a rich dynamical structure is revealed.

This rich dynamical structure we are referring to, corresponds to different systems which in our application are the Duffing oscillator (Heisenberg case), the simple

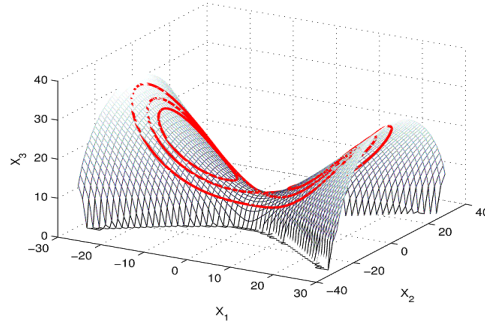


Figure 13. The phase-space of system (13) reduced on the hyperboloid C_0 for the case of AdS_2 .

pendulum ($SE(2)$ case), a uniformly charged rigid body in a uniform magnetic field ($SO(3)$ case) and a mathematical $SO(2,1)$ formulation. All these turn out to be different representations of the same dynamics! Similar analysis can be performed to any three dimensional volume preserving dynamical system providing insight on a unified description of dynamics for specific systems.

When the full dissipative system is considered the invariant manifolds of the conservative part can be used for the localization of the strange attractor. In addition, in section 4.1, an elementary analogue of a ‘spontaneous symmetry breaking mechanism’ is proposed as an intuitive explanation for the formation of the attractor. For the $SO(3)$ case, in section 4.3, the Lorenz system is physically identified with the rigid body system in a homogeneous magnetic field with external friction in every direction and a constant driven term in Z direction. A generalization of this rigid body system for arbitrary magnetic field’s direction leads to a dynamical system with new strange attractors (figures 11 and 12).

Last, we note that the $SO(3)$ formulation could be useful for an angular momentum or spin-based quantization of the Lorenz attractor in accordance with the matrix formulation of [6, 9]. It may also be interesting to investigate the correspondence with the $SO(2,1)$ picture in this quantum realm.

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Appendix A.

Applying the Lagrange’s multipliers method, we find the extremum of $S(X_1, X_2, X_3)$ under the constraint $\dot{S} = 0$ for:

$$S = (\lambda - 1)X_1^2 + X_2^2 + (X_3 - \lambda\rho)^2$$

and (X_1, X_2, X_3) subject to the Lorenz flow (7). Let call $A \equiv \dot{S}$.

We are looking for a point $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ and a Lagrange's multiplier α such, that: $K = S(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ is an extremum of S under the constraint $A = 0$. The constraint gives one equation:

$$-\sigma(\lambda - 1)\tilde{X}_1^2 - \tilde{X}_2^2 - b\tilde{X}_3^2 + b(\lambda\rho)\tilde{X}_3 = 0 \quad (\text{A.1})$$

Let extremize: $f = S + \alpha A$. From: $\nabla f = 0 \Rightarrow \nabla S + \alpha \nabla A = 0$ we get three more equations:

$$\tilde{X}_1(\lambda - 1)(1 - 2\alpha\sigma) = 0 \quad (\text{A.2})$$

$$\tilde{X}_2(1 - 2\lambda) = 0 \quad (\text{A.3})$$

$$\tilde{X}_3 = \lambda\rho \frac{1 - \alpha b}{1 - 2\alpha b} \quad (\text{A.4})$$

The system of Equations (A.1)-(A.4) has three distinct solutions:

- (i) $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = (0, 0, 0)$ and $\alpha = 1/b$
which gives $K = S(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = (\lambda\rho)^2$
- (ii) $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = (0, \sqrt{(\lambda\rho)^2 \frac{b^2(b-2)}{4(b-1)^2}}, \lambda\rho \frac{2-b}{2(1-b)})$ and $\alpha = 1/2$
which gives $K = S(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = (\lambda\rho)^2 \frac{b^2}{4(b-1)}$. From \tilde{X}_2 we see the constraint $b > 2$.
- (iii) $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = (\sqrt{(\lambda\rho)^2 \frac{b^2}{\sigma(1-\lambda)} \frac{(2\sigma-b)}{4\sigma(\sigma-b)^2}}, 0, \lambda\rho \frac{2\sigma-b}{2(\sigma-b)})$ and $\alpha = 1/2\sigma$
which gives $K = S(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = -(\lambda\rho)^2 \frac{b^2}{4\sigma(\sigma-b)}$. From \tilde{X}_1 we see the constraint $\lambda < 1$.

Appendix B.

Let a surface be defined by $S = K$ for some constant K and $S = S(\vec{X})$. If $A = \nabla S \cdot \vec{v}$ has the same sign at each and every point of S then S is a localization surface. Let calculate A for Lorenz flow \vec{v} (7) and $S = (\lambda - 1)X_1^2 + X_2^2 + (X_3 - \lambda\rho)^2$:

$$\begin{aligned} A \equiv \nabla S \cdot \vec{v} &= (2(\lambda - 1)X_1, 2X_2, 2(X_3 - \lambda\rho)) \\ &\quad * (\rho X_2 - \sigma X_1, -X_1 X_3 + \rho X_1 - X_2, X_1 X_2 - b X_3) \\ &= -2\sigma(\lambda - 1)X_1^2 - 2X_2^2 - 2bX_3^2 + 2b(\lambda\rho)X_3 \end{aligned} \quad (\text{B.1})$$

We will prove that A has a constant sign for all repelling surfaces of Equation (31).

- two sheeted hyperboloids $\lambda < 1$: $S = -R_r^2 \equiv -(\lambda\rho)^2 \frac{b^2}{4\sigma(\sigma-b)}$

Substituting: $(\lambda - 1)X_1^2 = -X_2^2 - X_3^2 + 2(\lambda\rho)X_3 - (\lambda\rho)^2 \frac{(2\sigma-b)^2}{4\sigma(\sigma-b)}$ in (B.1) we get:

$$\begin{aligned} A &= 2(\sigma - 1)X_2^2 + 2(\sigma - b)X_3^2 - 2(\lambda\rho)X_3(2\sigma - b) + 2\sigma(\lambda\rho)^2 \frac{(2\sigma - b)^2}{4\sigma(\sigma - b)} \\ &= 2(\sigma - 1)X_2^2 + \frac{1}{2(\sigma - b)} \\ &\quad \times \{4X_3^2(\sigma - b)^2 - 4(\lambda\rho)(2\sigma - b)X_3(\sigma - b) + (\lambda\rho)^2(2\sigma - b)^2\} \\ &= 2(\sigma - 1)X_2^2 + \frac{1}{2(\sigma - b)} \{2X_3(\sigma - b) - (\lambda\rho)(2\sigma - b)\}^2 > 0 \end{aligned}$$

$\forall (X_1, X_2, X_3) \in S$ and for $\sigma > 1, \sigma > b$.

- cone $\lambda = 0 : S = \tilde{R}_r^2 \equiv (\lambda\rho)^2 = 0$

Substituting: $X_1^2 = X_2^2 + X_3^2$ in (B.1) we get:

$$A = 2(\sigma - 1)X_2^2 + 2(\sigma - b)X_3^2 > 0$$

$\forall (X_1, X_2, X_3) \in S$ and for $\sigma > 1, \sigma > b$.

- parabolic surface $\lambda \rightarrow -\infty : S = \tilde{R}_r^2 \equiv (\lambda\rho)^2 \xrightarrow{\lambda \rightarrow -\infty} -X_1^2 + 2\rho X_3 = 0$

Substituting ρX_3 in (B.1) we get:

$$\begin{aligned} A &= (-X_1, 0, \rho) \cdot (\rho X_2 - \sigma X_1, -X_1 X_3 + \rho X_1 - X_2, X_1 X_2 - b X_3) \\ &= \sigma X_1^2 - b\rho X_3 = \sigma X_1^2 - \frac{b}{2} X_1^2 = (\sigma - \frac{b}{2}) X_1^2 > 0 \end{aligned}$$

$\forall (X_1, X_2, X_3) \in S$ and for $\sigma > b/2$.

- one sheeted hyperboloids $\lambda < 0 : S = \tilde{R}_r^2 \equiv (\lambda\rho)^2$

Substituting: $(\lambda - 1)X_1^2 = -X_2^2 - X_3^2 + 2(\lambda\rho)X_3$ in (B.1) we get:

$$2(\sigma - 1)X_2^2 + 2(\sigma - b)X_3^2 - 2(\lambda\rho)X_3(2\sigma - b) > 0$$

$\forall (X_1, X_2) \in S$ and $X_3 > 0$ and for $\sigma > 1, \sigma > b$. Only the part $X_3 > 0$ of the surface is a repelling surface. From the parabolic surface we know that the attractor lies in $X_3 > 0$. So these upper parts of one sheeted hyperboloids provide a lower boundary of the attractor.

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